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Diagonalization of the elliptic Ruijsenaars model. Correspondence with the Belavin model*

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Abstract. Studied is the elliptic Ruijsenaars model, which is a difference analogue of the Calogero-Sutherland-Moser model. Using a novel relationship between the elliptic Ruijsenaars operator and the transfer matrix of the Belavin model, we diagonalize the Ruijsenaars operator by the algebraic Bethe ansatz method.

PACS. 02.90.+p Other topics in mathematical methods in physics – 05.90.+m Other topics in statistical physics and thermodynamics – 75.10.Jm Quantized spin models

1 Introduction

The elliptic Ruijsenaars model was introduced as an integrable difference Schrödinger operator in [1]. This model can be regarded as an elliptic generalization of the Macdonald difference operator [2], and in the continuum limit it reduces to the elliptic Calogero-Sutherland-Moser (CSM) model [3]. The eigenfunction and the algebraic structure of the Macdonald operator have been recently well understood by use of the affine Hecke algebra [4–6], and from the physical view point the CSM model is described by quasi-particles which obey "exclusion statistics" [7]. On the other hand, studies of the elliptic Ruijsenaars model are not sufficient.

In this article as an extension of our previous works [8–10], we shall study a novel relationship between the elliptic Ruijsenaars model (with a specific coupling constant) and the inhomogeneous Belavin model. We show that, for a certain coupling constant, the elliptic Ruijsenaars operator preserves the finite dimensional function space, and that the Ruijsenaars difference operator is represented by use of the R-matrix for the Belavin model. Based on this representation, we shall apply the algebraic Bethe ansatz method to diagonalize the N-body elliptic Ruijsenaars operator.

This article is organized as follows. In Section 2 we review the construction of the elliptic Ruijsenaars operator. Our fundamental tool is Shibukawa-Ueno's elliptic R-operator which satisfies the Yang-Baxter equation. In Section 3 we show that the elliptic Ruijsenaars operator with a specific coupling constant corresponds to the transfer matrix of the inhomogeneous Belavin model. Diagonalization of the two-body problem is studied in Section 4. We take a trigonometric limit, and compare our results with previously known results. We then in Section 5 apply the algebraic Bethe ansatz method to derive eigenvalues of the Ruijsenaars operators. Last section is devoted to the concluding remarks.

2 Construction of the Ruijsenaars operator

We define the elliptic R-operator [11] as

$$R^{j,k}(\xi) = \frac{1}{\sigma_\mu(\xi;\tau)} \left(\sigma_\mu(z_{jk};\tau) - \sigma_\xi(z_{jk};\tau)\hat{s}_{j,k}\right), \qquad (2.1)$$

where ξ is called the spectral parameter, and μ is an arbitrary constant. For brevity we denote $z_{jk} = z_j - z_k$. Definitions and properties of the elliptic functions are summarized in Appendix. We have used the permutation operator $\hat{s}_{j,k}$ for coordinates z_j satisfying

$$\hat{s}_{j,k}^{\,2} = \mathbb{1}, \qquad z_j\,\hat{s}_{j,k} = \hat{s}_{j,k}\,z_k,$$

$$\hat{s}_{j,k}\,\hat{s}_{k,l} = \hat{s}_{k,l}\,\hat{s}_{l,j} = \hat{s}_{l,j}\,\hat{s}_{j,k},$$

where $j \neq k \neq l \neq j$. The R-operator (2.1) fulfills following relations;

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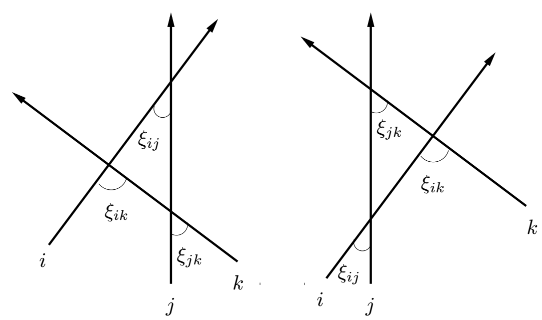
(A) the Yang-Baxter equation,

$$R^{i,j}(\xi_{ij})R^{i,k}(\xi_{ik})R^{j,k}(\xi_{jk}) = R^{j,k}(\xi_{jk})R^{i,k}(\xi_{ik})R^{i,j}(\xi_{ij}), \tag{2.2}$$



(B) the unitarity condition,

$$R^{j,k}(\xi)\, R^{k,j}(-\xi) = \mathbb{1}, \tag{2.3}$$

(C) the quasi-classical condition,

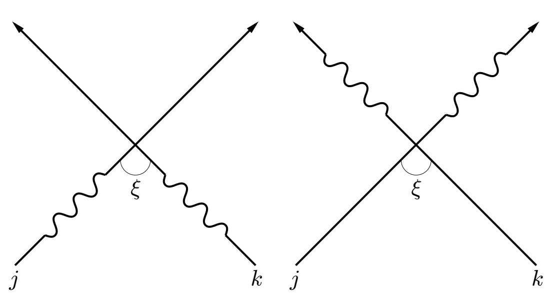
$$R^{j,k}(\xi = 0) = \hat{s}_{j,k}. \tag{2.4}$$

Besides the elliptic R-operator, we introduce the shift operator $\hat{T}_j(\beta)$ as

$$\left(\hat{T}_j(\beta)\, f\right)(z_1, \ldots, z_N) = f(z_1, \ldots, z_j + \beta, \ldots, z_N). \tag{2.5}$$

We note that the elliptic R-operator (2.1) commutes with the shift operators as

$$[R^{j,k}(\xi)\ ,\ \hat{T}_j(\beta)\,\hat{T}_k(\beta)] = 0. \tag{2.6}$$



To reduce the elliptic R-operator into a finite-dimensional matrix form, we define the vector space $V_m(\xi)$; a function $f(z) \in V_m(\xi)$ is an entire function, and satisfies doubly quasi-periodic conditions,

$$f(z+1) = f(z), \tag{2.7a}$$

$$f(z+\tau) = e^{-2\pi i m z - \pi i m \tau + 2\pi i \xi} f(z). \tag{2.7b}$$

The vector space $V_m(\xi)$ is m-dimensional, and its bases are written as

$$\begin{aligned}\theta_\alpha(z,\xi) &= \vartheta\begin{bmatrix}\frac{\alpha}{m}\\ -\xi\end{bmatrix}(mz; m\tau)\\ &= \sum_{n\in\mathbb{Z}} \exp\left(\pi i (nm+\alpha)^2 \frac{\tau}{m} + 2\pi i (nm+\alpha)\left(z - \frac{\xi}{m}\right)\right),\end{aligned} \tag{2.8}$$

where $\alpha \in \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. One sees that the shift operator $\hat{T}(-\xi/m)$ maps $V_m(0)$ isomorphically onto $V_m(\xi)$, and that the elliptic operator $R(\xi_{12})$ maps $V_m(\xi_1) \otimes V_m(\xi_2 + \mu)$ to $V_m(\xi_1 + \mu) \otimes V_m(\xi_2)$. In the following we use a notation θ_α as bases of $V_m(0)$.

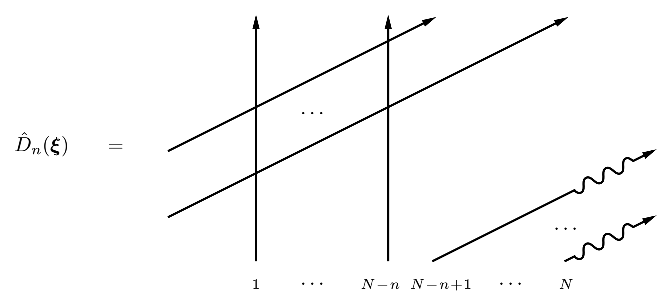
We shall now introduce the elliptic Ruijsenaars operator. By use of the R-operator (2.1), we define the difference operator as

$$\hat{D}_1(\boldsymbol{\xi}) = R^{N,N-1}(\xi_{NN-1}) \cdots R^{N,1}(\xi_{N1})\, \hat{T}_N(\beta), \tag{2.9}$$

where $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)$ and β are arbitrary constants. This operator $\hat{D}_1(\boldsymbol{\xi})$ is integrable [9,10], and the commutative difference operators, $[\hat{D}_{n_1}(\boldsymbol{\xi})\ ,\ \hat{D}_{n_2}(\boldsymbol{\xi})] = 0$ for $n_1, n_2 = 1, 2, \ldots N$, are defined as

$$\hat{D}_n(\boldsymbol{\xi}) = \prod_{j=N-n+1}^{N} \left(\prod_{k=1}^{\curvearrowleft\, N-n} R^{j,k}(\xi_{jk}) \right) \hat{T}_j(\beta). \tag{2.10}$$

It might be helpful to note that these difference operators $\hat{D}_n(\boldsymbol{\xi})$ can be depicted as



It should be remarked that we have another commutative operator $\hat{D}_0(\kappa)$,

$$\hat{D}_0(\kappa) = \prod_{j=1}^{N} \hat{T}_j(\kappa), \tag{2.11}$$

which denotes the total shift operator. As the difference operators $\hat{D}_n(\boldsymbol{\xi})$ are integrable for arbitrary $\boldsymbol{\xi}$, we set

$$\hat{D}_n = \lim_{\xi_{jk} = -(k-j)\mu} \hat{D}_n(\boldsymbol{\xi}). \tag{2.12}$$

It is known that the operators $\hat{D}_n$ preserve the symmetric function spaces. In fact, when we suppose that the operators $\hat{D}_n$ act on the symmetric z-space (*i.e.*, we replace the permutation operators $\hat{s}_{j,k}$ with unity when they are moved to the rightmost of expression), the difference operators $\hat{D}_n$ are explicitly written as [9]

$$\hat{D}_n = \prod_{k=1}^{n} \frac{\vartheta_1(k\mu;\tau)}{\vartheta_1((N+1-k)\mu;\tau)} \times \sum_{|I|=n} \left[\prod_{\substack{j\in I\\ k\in I^C}} \frac{\vartheta_1(z_{jk}-\mu;\tau)}{\vartheta_1(z_{jk};\tau)} \right] \hat{T}_I(\beta). \qquad (2.13)$$

This set of operators is nothing but the higher commuting Hamiltonian for the elliptic Ruijsenaars operator [1].

In the following we study the slightly modified difference operators defined by

$$\hat{\mathfrak{D}}_n = \hat{D}_n\, \hat{D}_0(-n\frac{\mu}{m}), \qquad (2.14)$$

with a condition (m is a positive integer),

$$\beta = N\frac{\mu}{m}\cdot \qquad (2.15)$$

Although the operators (2.14) are different from the original operators due to the commutative shift operator $\hat{D}_0(-n\mu/m)$, we call hereafter the difference operators $\hat{\mathfrak{D}}_n$ the elliptic Ruijsenaars operators. We should note that the lowest operator $\hat{\mathfrak{D}}_1$ is explicitly written as

$$\hat{\mathfrak{D}}_1 = \frac{\vartheta_1(\mu;\tau)}{\vartheta_1(N\mu;\tau)} \sum_{j=1}^{N} \left[\prod_{\substack{k=1\\ k\neq j}}^{N} \frac{\vartheta_1(z_{jk}-\mu;\tau)}{\vartheta_1(z_{jk};\tau)} \right] \times \hat{T}_j\left(\frac{N\mu}{m}\right) \left[\prod_{j=1}^{N} \hat{T}_j\left(-\frac{\mu}{m}\right) \right], \qquad (2.16)$$

which, in the trigonometric limit $e^{i\pi\tau}\to 0$, reduces to

$$\hat{\mathfrak{D}}_1^{\mathrm{Tri}} = \frac{\sin(\pi\mu)}{\sin(N\pi\mu)} \sum_{j=1}^{N} \left[\prod_{\substack{k=1\\ k\neq j}}^{N} \frac{\sin\pi(z_{jk}-\mu)}{\sin(\pi z_{jk})} \right] \times \hat{T}_j\left(\frac{N\mu}{m}\right) \left[\prod_{j=1}^{N} \hat{T}_j\left(-\frac{\mu}{m}\right) \right]. \qquad (2.17)$$

We remark that the two-body elliptic Ruijsenaars difference operator (2.16) with even m cases as the generalized Lamé equation has been also studied based on different techniques [12,13]. By use of the higher spin representation for the Sklyanin algebra [14] and the fusion procedure, the Bethe ansatz method is applied [12].

3 Correspondence with the Belavin model

We define the modified R-operator [15] as

$$R_m^{j,k}(\xi_{jk}) = \hat{T}_j\left(\frac{\xi_j+\mu}{m}\right) \hat{T}_k\left(\frac{\xi_k}{m}\right) R^{j,k}(\xi_{jk}) \times \hat{T}_j\left(-\frac{\xi_j}{m}\right) \hat{T}_k\left(-\frac{\xi_k+\mu}{m}\right), \qquad (3.1)$$

where m is a positive integer. One sees that the modified operator $R_m^{j,k}(\xi)$ is an endomorphism of the vector spaces $V_m(0)\otimes V_m(0)$. As far as we suppose that operators act on the vector space $V_m(0)^{\otimes N}$, the modified operator $R_m(\xi)$ is given by a finite dimensional representation as $m^2\times m^2$ matrix. We have [8,15,16]

$$R_m(\xi)\,\theta_\alpha\otimes\theta_\beta = \sum_{\gamma,\delta\in\mathbb{Z}_m} R_m(\xi)_{\alpha,\beta}^{\gamma,\delta}\,\theta_\gamma\otimes\theta_\delta, \qquad (3.2)$$

where the matrix elements $R_m(\xi)_{\alpha,\beta}^{\gamma,\delta}$ are computed as

$$R_m(\xi)_{\alpha,\beta}^{\gamma,\delta} = \frac{1}{\sigma_\mu(\xi;\tau)} \times \frac{\vartheta'\begin{bmatrix}\frac{1}{2}\\ \\ \frac{1}{2}\end{bmatrix}(0;m\tau)\,\vartheta\begin{bmatrix}\frac{1}{2}-\frac{\alpha-\beta}{m}\\ \\ \frac{1}{2}\end{bmatrix}(\xi-\mu;m\tau)}{\vartheta\begin{bmatrix}\frac{1}{2}-\frac{\gamma-\beta}{m}\\ \\ \frac{1}{2}\end{bmatrix}(\xi;m\tau)\,\vartheta\begin{bmatrix}\frac{1}{2}-\frac{\alpha-\gamma}{m}\\ \\ \frac{1}{2}\end{bmatrix}(-\mu;m\tau)}, \qquad (3.3)$$

iff $\alpha+\beta=\gamma+\delta \bmod m$, or else vanish. This R-matrix is nothing but the Boltzmann weight for the Belavin model [17,18], and coincides with Baxter's eight-vertex model [19] in the case of $m=2$. In the following we set the vector space as $V_m(0)^{\otimes N}$, *i.e.*, the modified elliptic R-operator can be identified with the Belavin R-matrix.

As the modified R-operator $R_m(\xi)$ also satisfies relations (2.2–2.4), we can define the integrable lattice model in the usual way. The transfer matrix for the inhomogeneous Belavin model is given by

$$T_N(\lambda;\boldsymbol{\xi}) = \mathrm{Tr}_a\, R_m^{a,N}(\lambda-\xi_N)\,\cdots\,R_m^{a,1}(\lambda-\xi_1), \qquad (3.4)$$

where a is an auxiliary space, and parameters $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_N)$ denotes the inhomogeneity. Main observation in this article is that the difference Ruijsenaars operator $\hat{\mathfrak{D}}_1$ (2.16) is written as

$$\begin{aligned}\hat{\mathfrak{D}}_1 &= R_m^{N,N-1}(-\mu)\,\cdots\,R_m^{N,1}(-(N-1)\mu)\\ &= \lim_{\substack{\lambda=\xi_N\\ \xi_{jk}=-(j-k)\mu}} T_N(\lambda;\boldsymbol{\xi}).\end{aligned} \qquad (3.5)$$

This equality indicates that the Ruijsenaars operator $\hat{\mathfrak{D}}_1$ preserves a function space $V_m(0)^{\otimes N}$. In the same manner, one sees that the higher commuting Hamiltonian operators $\hat{\mathfrak{D}}_n$ (2.14) also preserve a space $V_m(0)^{\otimes N}$. We can therefore diagonalize the transfer matrix of the Belavin model to calculate the eigenvalues of the elliptic Ruijsenaars model $\hat{\mathfrak{D}}_1$ on $V_m(0)^{\otimes N}$.

4 Two-body problem

We consider a direct diagonalization of the two-body ($N = 2$) Ruijsenaars operator (2.16). This difference operator has also appeared in the representation of the Sklyanin algebra [14]. In this case the eigenvalues are given by diagonalizing the $m^2 \times m^2$ matrix $R_m(-\mu)$ (3.3). Note that the eigenvalues for $\frac{1}{2}m(m-1)$ anti-symmetric functions are zeros, and that we should pick up $\frac{1}{2}m(m+1)$ symmetric functions as eigenfunctions of the two-body Ruijsenaars operator (2.16). We do not know the explicit form of the eigenvalues for general m cases, and in the following we only give some examples and their trigonometric limit. As we have diagonalized the elliptic Ruijsenaars operator $\hat{\mathfrak{D}}_1$ only on $V_m(0)^{\otimes N}$, we recover a part of the Macdonald polynomials in the trigonometric limit.

(1) $m = 2$,

Eigenvalues	Eigenstates
$2R_2(-\mu)_{01}^{01}$	$\mid\uparrow\rangle_1 \mid\downarrow\rangle_2 + \mid\downarrow\rangle_1 \mid\uparrow\rangle_2$
$R_2(-\mu)_{00}^{00} + R_2(-\mu)_{00}^{11}$	$\mid\uparrow\rangle_1 \mid\uparrow\rangle_2 + \mid\downarrow\rangle_1 \mid\downarrow\rangle_2$
$R_2(-\mu)_{00}^{00} - R_2(-\mu)_{00}^{11}$	$\mid\uparrow\rangle_1 \mid\uparrow\rangle_2 - \mid\downarrow\rangle_1 \mid\downarrow\rangle_2$

where we set spin states as

$$|\uparrow\rangle_j = \vartheta_3(2z_j; 2\tau), \qquad |\downarrow\rangle_j = \vartheta_2(2z_j; 2\tau).$$

In the trigonometric limit $e^{i\pi\tau} \to 0$, above result reduces into

Eigenvalues	Eigenstates
1	1
$\dfrac{1}{\cos(\pi\mu)}$	$\sin(2\pi z_1) + \sin(2\pi z_2)$,
	$\cos(2\pi z_1) + \cos(2\pi z_2)$

(2) $m = 3$,

Eigenvalues	Eigenstates
$\frac{1}{2}(a+b+c-\Delta)$	$\mid\uparrow\rangle_1 \mid 0\rangle_2 + \mid 0\rangle_1 \mid\uparrow\rangle_2 + A_- \mid\downarrow\rangle_1 \mid\downarrow\rangle_2$
	$A_- \mid\uparrow\rangle_1 \mid\uparrow\rangle_2 + \mid 0\rangle_2 \mid\downarrow\rangle_2 + \mid\downarrow\rangle_1 \mid 0\rangle_2$
	$A_- \mid 0\rangle_1 \mid 0\rangle_2 + \mid\uparrow\rangle_1 \mid\downarrow\rangle_2 + \mid\downarrow\rangle_1 \mid\uparrow\rangle_2$
$\frac{1}{2}(a+b+c+\Delta)$	$\mid\uparrow\rangle_1 \mid 0\rangle_2 + \mid 0\rangle_1 \mid\uparrow\rangle_2 + A_+ \mid\downarrow\rangle_1 \mid\downarrow\rangle_2$
	$A_+ \mid\uparrow\rangle_1 \mid\uparrow\rangle_2 + \mid 0\rangle_1 \mid\downarrow\rangle_2 + \mid\downarrow\rangle_1 \mid 0\rangle_2$
	$A_+ \mid 0\rangle_1 \mid 0\rangle_2 + \mid\uparrow\rangle_1 \mid\downarrow\rangle_2 + \mid\downarrow\rangle_1 \mid\uparrow\rangle_2$

where each parameter is defined as

$$A_\pm = \frac{a-b-c \pm \Delta}{2h}, \quad \Delta = \sqrt{(a-b-c)^2 + 4(f+g)h},$$

$$\begin{aligned} a &= R_3(-\mu)_{00}^{00}, \quad & b &= R_3(-\mu)_{01}^{01}, \quad & c &= R_3(-\mu)_{02}^{02}, \\ f &= R_3(-\mu)_{12}^{00}, \quad & g &= R_3(-\mu)_{21}^{00}, \quad & h &= R_3(-\mu)_{00}^{12}. \end{aligned}$$

Each spin state is given as

$$|\uparrow\rangle_j = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (3z_j; 3\tau),$$

$$|0\rangle_j = \vartheta \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} (3z_j; 3\tau),$$

$$|\downarrow\rangle_j = \vartheta \begin{bmatrix} 2/3 \\ 0 \end{bmatrix} (3z_j; 3\tau).$$

In a trigonometric limit, we have

Eigenvalues	Eigenstates
1	$1, \quad \sin 2\pi(z_1 + z_2)$,
	$\cos 2\pi(z_1 + z_2)$,
$\dfrac{\cos\left(\frac{1}{3}\pi\mu\right)}{\cos(\pi\mu)}$	$\cos(2\pi z_1) + \cos(2\pi z_2)$,
	$\sin(2\pi z_1) + \sin(2\pi z_2)$,
	$\cos(2\pi z_{12}) + \dfrac{\cos(\frac{2}{3}\pi\mu)\sin(\pi\mu)}{\sin(\frac{1}{3}\pi\mu)}$

5 Bethe Ansatz

It is not straightforward to diagonalize the N-body Ruijsenaars operator (2.16) as in the previous section. Owing to the correspondence (3.5) between the Ruijsenaars operator and the transfer matrix of the inhomogeneous Belavin model, we can apply the Bethe ansatz method.

5.1 Eight-Vertex model

We first consider the N-body problem with $m = 2$. We diagonalize the transfer matrix of the inhomogeneous XYZ model following the usual algebraic Bethe ansatz method [20–22], and then using the correspondence (3.5) we compute the eigenvalues of the Ruijsenaars operator (2.16). As was demonstrated in the previous section, the up- and down-spin for the XYZ spin chain is replaced by the elliptic functions $\vartheta_3(2z; 2\tau)$ and $\vartheta_2(2z; 2\tau)$, respectively. When we set $M = N/2$, the eigenvalue $T_N(\lambda; \boldsymbol{\xi})$ of the transfer matrix $\hat{T}_N(\lambda; \boldsymbol{\xi})$ is given as

$$\begin{aligned} T_N(\lambda; \boldsymbol{\xi}) = & \prod_{j=1}^{N} \frac{\sigma_\mu(\lambda - \xi_j; 2\tau)}{\sigma_\mu(\lambda - \xi_j; \tau)} \\ & \times \Bigg(e^{2\pi i\theta} \prod_{l=1}^{N} h_+(\lambda - \xi_l) \prod_{k=1}^{M} \alpha(\lambda, \lambda_k) \\ & + e^{-2\pi i\theta} \prod_{l=1}^{N} h_-(\lambda - \xi_l) \prod_{k=1}^{M} \alpha(\lambda_k, \lambda) \Bigg), \end{aligned} \tag{5.1}$$

where functions are defined as follows;

$$h_+(\xi) = \frac{\vartheta_0(0;2\tau)\,\vartheta_0(\xi-\mu;2\tau)}{\vartheta_0(-\mu;2\tau)\,\vartheta_0(\xi;2\tau)},$$

$$h_-(\xi) = \frac{\vartheta_0(0;2\tau)\,\vartheta_1(\xi;2\tau)}{\vartheta_0(-\mu;2\tau)\,\vartheta_1(\xi-\mu;2\tau)},$$

$$\alpha(\xi_1,\xi_2) = \frac{\vartheta_0(\xi_{21}-\mu;2\tau)\vartheta_1(\xi_{21}-\mu;2\tau)}{\vartheta_0(\xi_{21};2\tau)\vartheta_1(\xi_{21};2\tau)}\,.$$

Rapidities λ_k and θ are solutions of the Bethe ansatz equations,

$$\prod_{k=1}^{N}\frac{h_+(\lambda_j-\xi_k)}{h_-(\lambda_j-\xi_k)} = e^{-4\pi i\theta}\prod_{\substack{k=1\\k\neq j}}^{M}\frac{\alpha(\lambda_k,\lambda_j)}{\alpha(\lambda_j,\lambda_k)}\,. \tag{5.2}$$

Note that the Bethe eigenstates are defined as a Fourier series,

$$\Psi_\theta(\boldsymbol{\lambda}) = \sum_{l=-\infty}^{\infty} e^{2\pi i l\theta}\, B_{l+1,l-1}(\lambda_1)\cdots B_{l+M,l-M}(\lambda_M)|\Omega^{l-M}\rangle, \tag{5.3}$$

where operators $B_{k,l}(\lambda)$ are the Bethe creation operator, and the pseudo-vacuum $|\Omega^l\rangle = \omega_1^l\otimes\cdots\otimes\omega_N^l$ is given with arbitrary parameter s as

$$\begin{aligned}\omega_n^l &= \vartheta_1(s-\xi_n-(n+l-\frac{1}{2})\mu;2\tau)\\ &\quad\times|\uparrow\rangle + \vartheta_0(s-\xi_n-(n+l-\frac{1}{2})\mu;2\tau)|\downarrow\rangle.\end{aligned}$$

To obtain the eigenvalue of the elliptic Ruijsenaars operator $\hat{\mathcal{D}}_1$, we take a limit (3.5), $\lambda=\xi_N$ and $\xi_{jk}=-(j-k)\mu$, in (5.1, 5.2), and use the Landen transformation formula. As a result, we obtain the eigenvalues of the elliptic Ruijsenaars operator (2.16) as

$$\mathcal{D}_1 = e^{2\pi i\theta}\prod_{l=1}^{M}\frac{\vartheta_1(\tilde{\lambda}_l-\mu;\tau)}{\vartheta_1(\tilde{\lambda}_l;\tau)}, \tag{5.4}$$

where rapidities $\tilde{\lambda}_k \equiv \lambda_k+\xi_N$ and θ satisfy the Bethe ansatz equation,

$$\prod_{\substack{k=1\\k\neq j}}^{M}\frac{\vartheta_1(\tilde{\lambda}_{jk}-\mu;\tau)}{\vartheta_1(\tilde{\lambda}_{jk}+\mu;\tau)} = e^{4\pi i\theta}\frac{\vartheta_1(\tilde{\lambda}_j-N\mu;\tau)}{\vartheta_1(\tilde{\lambda}_j;\tau)}, \tag{5.5}$$

for $j=1,2,\ldots,M=\frac{N}{2}$. We further set the eigenstates $\Psi_\theta(\boldsymbol{\lambda})$ to be the symmetric functions, though unfortunately it is generally not trivial to write the explicit form of the Bethe states (5.3).

5.2 Belavin model

In the same way, we can apply the (nested) Bethe ansatz for general m case. Following a result of [23], we obtain the eigenvalue of the elliptic Ruijsenaars operator (2.16) as

$$\mathcal{D}_1 = e^{i\langle\bar{\epsilon}_0,\theta\rangle}\prod_{l=1}^{M_1}\frac{\vartheta_1(\lambda_l^{(1)}-\mu;\tau)}{\vartheta_1(\lambda_l^{(1)};\tau)}\,. \tag{5.6}$$

Here the vectors $\bar{\epsilon}_\alpha$ for $\alpha=0,1,\ldots,m-1$ and θ are given using a orthonormal vector ϵ_α as

$$\bar{\epsilon}_\alpha = \epsilon_\alpha - \frac{1}{m}\sum_{\beta=0}^{m-1}\epsilon_\beta, \qquad \theta = \sum_{\beta=0}^{m-1}\theta_\beta\,\epsilon_\beta.$$

Parameters M_α and spectral parameters $\lambda_j^{(0)}$ are given by

$$M_\alpha = \Big(1-\frac{\alpha}{m}\Big)N, \quad \lambda_j^{(0)} = -(j-N)\mu.$$

Rapidities $\lambda_j^{(\alpha)}$ and θ_β satisfy the nested Bethe ansatz equations,

$$\begin{aligned}&e^{i\langle\bar{\epsilon}_\alpha-\bar{\epsilon}_{\alpha+1},\theta\rangle}\prod_{l=1}^{M_\alpha}\frac{\vartheta_1(\lambda_j^{(\alpha+1)}-\lambda_l^{(\alpha)}-\mu)}{\vartheta_1(\lambda_j^{(\alpha+1)}-\lambda_l^{(\alpha)})}\\ &\quad=\prod_{\substack{k=1\\k\neq j}}^{M_{\alpha+1}}\frac{\vartheta_1(\lambda_{jk}^{(\alpha+1)}-\mu)}{\vartheta_1(\lambda_{jk}^{(\alpha+1)}+\mu)}\prod_{l=1}^{M_{\alpha+2}}\frac{\vartheta_1(\lambda_l^{(\alpha+2)}-\lambda_j^{(\alpha+1)}-\mu)}{\vartheta_1(\lambda_l^{(\alpha+2)}-\lambda_j^{(\alpha+1)})}\,.\end{aligned} \tag{5.7}$$

As was noticed in the eight-vertex model, we should pick up the symmetric eigenstates.

6 Conclusion

We have shown a novel relationship between the elliptic Ruijsenaars operator and the Belavin model. The transfer matrix of the inhomogeneous Belavin model coincides with the Ruijsenaars operator with a specific coupling constant over function space $V_m(0)^{\otimes N}$. For simple cases such as two-body problem, we have checked that our solutions give previously known Macdonald polynomials in the trigonometric limit. We should stress that, contrary to the case of the Belavin model, the space $V_m(0)^{\otimes N}$ is not complete as a function space of the Ruijsenaars operator (2.16). See [24] for discussions on complete function space. We have also derived the eigenvalues of the N-body elliptic Ruijsenaars operator (2.16) by use of the Bethe ansatz.

Our method will be applied to the generalized elliptic Macdonald-Koornwinder operator (the D-type Macdonald operator) [9,25] when we consider the Belavin model with open boundary. We hope to study this problem in future issues.

Appendix: Elliptic function

We define the theta functions as [26],

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z;\tau) = \sum_{n\in\mathbb{Z}} \exp\big(\pi i(n+a)^2\tau + 2\pi i(n+a)(z+b)\big)\,, \quad \text{(A.1)}$$

where $\Im\tau > 0$. The zeros of the theta function $\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z;\tau)$ are $z = (1/2 - a)\,\tau + (1/2 - b)$ modulo $\mathbb{Z} + \mathbb{Z}\tau$. We also use the following notations;

$$\vartheta_1(z;\tau) = \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z;\tau), \quad \vartheta_2(z;\tau) = \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (z;\tau),$$
$$\vartheta_3(z;\tau) = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z;\tau), \qquad \vartheta_0(z;\tau) = \vartheta_4(z;\tau) = \vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (z;\tau).$$

In terms of these theta functions, we define $\sigma_\mu(\xi;\tau)$ as

$$\sigma_\mu(\xi;\tau) = \frac{\vartheta_1'(0;\tau)\,\vartheta_1(\xi-\mu;\tau)}{\vartheta_1(\xi;\tau)\,\vartheta_1(-\mu;\tau)}, \quad \text{(A.2)}$$

where $\vartheta_1'(\xi;\tau)$ denotes a derivative with respect to ξ. By comparing zeros and poles of the both hands sides, we can derive useful formulae such as

$$\begin{aligned} \sigma_\lambda(z;\tau)\,\sigma_\mu(w;\tau) &= \sigma_{\lambda+\mu}(w;\tau)\,\sigma_\lambda(z-w;\tau) \\ &\quad + \sigma_\mu(w-z;\tau)\,\sigma_{\lambda+\mu}(z;\tau), \\ \sigma_\mu(z;\tau) &= \sum_{k=0}^{m-1} \sigma_{\mu-k\tau}(mz;m\tau)\,e^{2\pi i k z}, \\ \prod_{k=0}^{m-1} \vartheta \begin{bmatrix} \frac{1}{2}+\frac{k}{m} \\ \frac{1}{2} \end{bmatrix} (z;m\tau) &= \gamma_0\,\vartheta_1(z;\tau), \end{aligned}$$

where γ_0 does not depend on z.

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